MTH 166

Lecture-17

Euler-Cauchy Equation

Topic:

Solution of Non-Homogeneous LDE with Variable coefficients.

Learning Outcomes:

1. To convert equation with variable coefficients (Euler-Cauchy Equation) to an equation with constant coefficients and then solve it with standard known methods.

2. To write operator form of simultaneous system of LDE

Euler-Cauchy Equation:

Let us consider 2nd order Non-homogeneous LDE with variable coefficients as:

$$x^2y'' + xy' + y = r(x) \qquad (1)$$

Equation of type (1) called Euler-Cauchy Equation.

S.F. : $(x^2D^2 + xD + 1)y = r(x)$ (2) where $D \equiv \frac{d}{dx}$

The very first job is to convert this equation with variable coefficients to an equation with constant coefficients using an appropriate transformation.

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

Let $\theta \equiv \frac{d}{dt}$ (Another differential operator)

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1),\ x^3D^3 = \theta(\theta - 1)(\theta - 2)$ and so on…

Equation (2) becomes: $[\theta(\theta - 1) + \theta + 1]y = r(t)$ (3)

which is an equation with constant coefficients and we know the methods to solve equ.(3)

Problem 1. Find the general solution of: $x^2y'' + xy' - 4y = 0$

Solution: The given equation is:

$x^2y'' + xy' - 4y = 0$ (1)

S.F. : $(x^2D^2 + xD - 4)y = 0$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta, \; x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta - 1) + \theta - 4]y = 0$

A.E. : $[\theta(\theta - 1) + \theta - 4] = 0 \quad \Rightarrow [\theta^2 - \theta + \theta - 4] = 0$

$\Rightarrow (\theta^2 - 4) = 0 \quad \Rightarrow \theta = 2, -2$ (real and unequal roots)

$\therefore$ General solution is given by:

$\Rightarrow y_c = c_1e^{2t} + c_2e^{-2t} \quad \Rightarrow y_c = c_1x^2 + c_2x^{-2}$ **Answer.**

Problem 2. Find the general solution of: $9x^2y'' + 15xy' + y = 0$

Solution: The given equation is:

$9x^2y'' + 15xy' + y = 0 \qquad (1)$

S.F. : $(9x^2D^2 + 15xD + 1)y = 0 \qquad (2) \qquad$ where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1) \qquad$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[9\theta(\theta - 1) + 15\theta + 1]y = 0$

A.E. : $[9\theta(\theta - 1) + 15\theta + 1] = 0 \quad \Rightarrow [9\theta^2 - 9\theta + 15\theta + 1] = 0$

$\Rightarrow (9\theta^2 + 6\theta + 1) = 0 \quad \Rightarrow (3\theta + 1)(3\theta + 1) = 0 \quad \Rightarrow \theta = -\frac{1}{3}, -\frac{1}{3}$ (equal roots)

$\therefore$ General solution is given by:

$\Rightarrow y_c = (c_1 + c_2t)e^{-\frac{1}{3}t} \quad \Rightarrow y_c = (c_1 + c_2 \log x)\, x^{-\frac{1}{3}}$ **Answer.**

Problem 3. Find the general solution of: $2x^2y'' + 2xy' + y = 0$

Solution: The given equation is:

$2x^2y'' + 2xy' + y = 0$ (1)

S.F. : $(2x^2D^2 + 2xD + 1)y = 0$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[2\theta(\theta - 1) + 2\theta + 1]y = 0$

A.E. : $[2\theta(\theta - 1) + 2\theta + 1] = 0 \quad \Rightarrow [2\theta^2 - 2\theta + 2\theta + 1] = 0$

$\Rightarrow (2\theta^2 + 1) = 0 \quad \Rightarrow \theta^2 = -\frac{1}{2} \quad \Rightarrow \theta = \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}}i$ (complex roots)

$\therefore$ General solution is given by:

$\Rightarrow y_c = e^{0t}(c_1 \cos\frac{1}{\sqrt{2}}t + c_2 \sin\frac{1}{\sqrt{2}}\log x) \quad \Rightarrow y_c = (c_1 \cos\frac{1}{\sqrt{2}}\log x + c_2 \sin\frac{1}{\sqrt{2}}\log x)$ **Ans.**

Problem 4. Find the general solution of: $x^2y'' - 2y = 2x$

Solution: The given equation is:

$x^2y'' - 2y = 2x$ (1)

S.F. : $(x^2D^2 - 2)y = 2x$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta - 1) - 2]y = 2e^t$

$\Rightarrow f(\theta)y = r(t)$ where $f(\theta) = (\theta^2 - \theta - 2)$ and $r(t) = 2e^t$

To find Complimentary Function (C.F.):

A.E. : $f(\theta) = 0 \quad \Rightarrow (\theta^2 - \theta - 2) = 0 \quad \Rightarrow (\theta - 2)(\theta + 1) = 0$

$\Rightarrow \theta = 2, -1$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1e^{2t} + c_2e^{-t} \quad \Rightarrow y_c = c_1x^2 + c_2x^{-1}$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(\theta)} r(t) = \frac{1}{(\theta^2 - \theta - 2)} (2e^t)$$

$$\Rightarrow y_p = 2\left[\frac{1}{(\theta^2 - \theta - 2)} e^t\right] \qquad \Rightarrow y_p = 2\left[\frac{1}{\left((1)^2 - (1) + 2\right)} e^t\right] \quad (Put\ D = 2)$$

$$\Rightarrow y_p = 2\left[\frac{1}{2} e^t\right] \qquad \Rightarrow y_p = e^t = x$$

∴ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 x^2 + c_2 x^{-1}) + x$ **Answer.**

Problem 5. Find the general solution of: $x^2y'' + 2xy' = \cos(\log x)$

Solution: The given equation is:

$x^2y'' + 2xy' = \cos(\log x)$ (1)

S.F. : $(x^2D^2 + 2xD)y = \cos(\log x)$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta - 1) + 2\theta]y = \cos t$

$\Rightarrow f(\theta)y = r(t)$ where $f(\theta) = (\theta^2 + \theta)$ and $r(t) = \cos t$

To find Complimentary Function (C.F.):

A.E. : $f(\theta) = 0 \quad \Rightarrow (\theta^2 + \theta) = 0 \quad \Rightarrow \theta(\theta + 1) = 0$

$\Rightarrow \theta = 0, -1$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1e^{0t} + c_2e^{-t} \quad \Rightarrow y_c = c_1 + c_2x^{-1}$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(\theta)} r(t) = \frac{1}{(\theta^2+\theta)} (\cos t)$$

$$\Rightarrow y_p = \left[\frac{1}{(-(1)^2+\theta)} (\cos t)\right] \qquad (Put\ \theta^2 = -(1)^2)$$

$$\Rightarrow y_p = \left[\frac{1}{(\theta-1)} (\cos t)\right] = \left[\frac{1}{(\theta-1)} \times \frac{(\theta-1)}{(\theta-1)} (\cos t)\right] = \left[\frac{(\theta-1)}{(\theta^2-1)} (\cos t)\right]$$

$$\Rightarrow y_p = \left[\frac{(\theta-1)}{(-1-1)} (\cos t)\right] = -\frac{1}{2}\left[\frac{d}{dt} (\cos t) - (\cos t)\right] = \frac{1}{2} (\sin t + \cos t)$$

$$\Rightarrow y_p = \frac{1}{2} [\sin(\log x) + \cos(\log x)]$$

∴ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$$\Rightarrow y = c_1 + c_2 x^{-1} + \frac{1}{2} [\sin(\log x) + \cos(\log x)]$$ **Answer.**

Problem 6. Find the general solution of: $x^2y'' - 3xy' + 3y = 2 + 3\log x$

Solution: The given equation is:

$$x^2y'' - 3xy' + 3y = 2 + 3\log x \qquad (1)$$

S.F. : $(x^2D^2 - 3xD + 3)y = 2 + 3\log x$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta - 1) - 3\theta + 3]y = 2 + 3t$

$\Rightarrow f(\theta)y = r(t)$ where $f(\theta) = (\theta^2 - 4\theta + 3)$ and $r(t) = 2 + 3t$

To find Complimentary Function (C.F.):

A.E. : $f(\theta) = 0 \quad \Rightarrow (\theta^2 - 4\theta + 3) = 0 \quad \Rightarrow (\theta - 1)(\theta - 3) = 0$

$\Rightarrow \theta = 1, 3$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1e^t + c_2e^{3t} \quad \Rightarrow y_c = c_1x + c_2x^3$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(\theta)} r(t) = \frac{1}{(\theta^2-4\theta+3)}(2+3t)$$

$$\Rightarrow y_p = \left[\frac{1}{3\left[1+\left(\frac{\theta^2-4\theta}{3}\right)\right]}(2+3t)\right] = \frac{1}{3}\left[\left(1+\left(\frac{\theta^2-4\theta}{3}\right)\right)^{-1}(2+3t)\right]$$

$$\Rightarrow y_p = \frac{1}{3}\left[\left(1-\left(\frac{\theta^2-4\theta}{3}\right)+\left(\frac{\theta^2-4\theta}{3}\right)^2 ---\right)(2+3t)\right]$$

$$\Rightarrow y_p = \frac{1}{3}\left[(2+3t)+\frac{4}{3}\frac{d}{dt}(2+3t)+0\right] = \frac{1}{3}\left[(2+3t)+4\right] = 2+t$$

$$\Rightarrow y_p = 2+\log x$$

$\therefore$ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = c_1 x + c_2 x^3 + 2 + \log x$ **Answer.**

<u>Simultaneous Linear Differential Equations:</u>

The system involving two first order linear differential equations in two dependent variables y_1 and y_2 and one independent variable x is called system of simultaneous linear differential equations.

From MCQ point of view, we will just write the operator form of simultaneous equations. Their solution becomes quite lengthy and has no scope for MCQ pattern.

Write the operator form of following system of LDE:

Problem1. $6\frac{dy_1}{dx}+5\frac{dy_2}{dx}+3y_1+y_2=0, \frac{dy_2}{dx}-5y_1+3y_2=e^x$

Solution. The given system of simultaneous equations is:

$$6\frac{dy_1}{dx}+5\frac{dy_2}{dx}+3y_1+y_2=0 \qquad (1)$$

$$\frac{dy_2}{dx}-5y_1+3y_2=e^x \qquad (2)$$

Let $D\equiv\frac{d}{dx}$, then operator form of given system can be written as:

$$(6D+3)y_1+(5D+1)y_2=0 \qquad (3)$$

$$-5y_1+(D+3)y_2=e^x \qquad (4)$$

Problem2. $3\frac{dy_1}{dx} + 2y_1 + y_2 = e^{-x}, \frac{dy_1}{dx} + \frac{dy_2}{dx} - 2y_1 + 3y_2 = x$

Solution. The given system of simultaneous equations is:

$$3\frac{dy_1}{dx} + 2y_1 + y_2 = e^{-x} \qquad (1)$$

$$\frac{dy_1}{dx} + \frac{dy_2}{dx} - 2y_1 + 3y_2 = x \qquad (2)$$

Let $D \equiv \frac{d}{dx}$, then operator form of given system can be written as:

$$(3D + 2)y_1 + y_2 = e^{-x} \qquad (3)$$

$$(D - 2)y_1 + (D + 3)y_2 = x \qquad (4)$$